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Propagation of Cladded Inhomogeneous Dielectric Waveguides

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Abstract—**An approximate theory on the propagation of modes in an arbitrarily inhomogeneous optical waveguide embedded in a homogeneous medium is presented. Simple formulas are given, whereby the propagation constants can be determined assuming that the analytic solution is known in the absence of cladding. The results obtained applying the theory to a truncated parabolic-index profile are shown to be in good agreement with those obtained by the rigorous analysis. The theory is also applied to the propagation of TE and TM waves in truncated near-parabolic-index media.**

Introduction

An important aspect of guided waves propagating in inhomogeneous (graded) index dielectric waveguides involves the investigation of the propagation characteristics subjected to signal distortion. Recently, considerable effort has been expended to compute the propagation constants of modes by means of high-accuracy straightforward computation [1], because it is very difficult to obtain an analytic solution except for a certain index profile. The analytic approach becomes more difficult for the cladded inhomogeneous dielectric waveguide in which the graded-index medium is suspended in a homogeneous medium.

In a recent work [2], the purely mathematical techniques, based on the integral representation of a solution of Hermite's differential equation, have successfully been applied to a cladded parabolic-index waveguide, and the mode functions have been obtained in analytic form.

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The aim of this paper is to develop an approximate theory of propagating modes in a general class of cladded inhomogeneous dielectric waveguides. This theory is verified by comparing the results with those obtained exactly in a case of parabolic-index profile.

We here consider the two-dimensional waveguide in which the refractive index varies in the transverse x direction as (see Fig. 1)

$$n(x) = \begin{cases} n_0 \sqrt{1 - \chi(x)}, & \text{for } |x| < x_c \\ n_0 \sqrt{1 - \chi(x_c)}, & \text{for } |x| > x_c \end{cases} \quad (1)$$

where $\chi(x)$ is an even and a smooth function satisfying $\chi(0) = 0$ (see Fig. 1), and n_0 is the refractive index at the center axis z ($x = 0$). The lower order modes are allowed to propagate along the z axis in the guiding medium $|x| < x_c$, and the undesirable higher order modes are radiated through the homogeneous outer medium $|x| > x_c$. The present approximate theory is developed for the TE wave propagation along such a waveguide. However, it is shown that the theory can be extended to the problem of the TM wave propagation. An example of determining the propagation constants of TM waves in truncated near-parabolic-index media is given in the last subsection.

Modes in an Ideal Waveguide

The starting point is the knowledge of modes (electric-field functions), including propagation constants, in the ideal waveguide, which is defined as an uncladded waveguide consisting only of the guiding material (the index distribution is indicated by the dotted curve in Fig. 1).

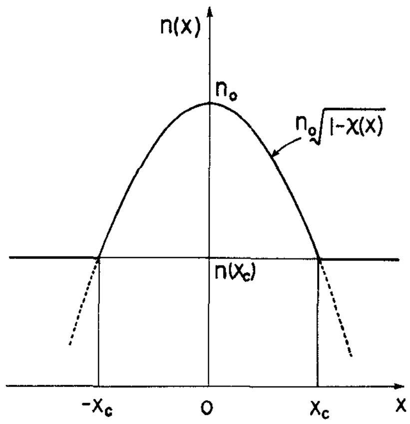

Fig. 1. Refractive-index distribution of a cladded inhomogeneous dielectric waveguide (solid curve). The dotted curve indicates the refractive-index distribution of an ideal waveguide (an uncladded inhomogeneous dielectric waveguide).

We now write the mode function and the propagation constant in the ideal waveguide as

$$\Phi_n(x)e^{-j\beta_n z} \qquad \beta_n = k\sqrt{1-b_n}, \qquad (n = 0,1,2,\cdots) \quad (2)$$

where k is the wavenumber at the center axis z. The transverse mode function $\Phi_n(x)$ obeys the one-dimensional wave equation

$$\Phi_n''(x) + k^2[b_n - \chi(x)]\Phi_n(x) = 0 \quad (3)$$

where primes denote differentiation with respect to x.

A. Acceptable Solution

To obtain the guided wave along the center axis z, the transverse mode function $\Phi_n(x)$ must be assumed to be a damped solution that decreases exponentially as x tends to infinity. When x is restricted to the oscillatory region in which the field behaves oscillatorily, the solution can be represented in the WKB asymptotic form [3]

$$\Phi_n(x) \simeq C_n \frac{\sqrt{2Z_0}}{\sqrt[4]{b_n - \chi(x)}} \cos\left(\int_0^x k\sqrt{b_n - \chi(t)}\, dt - \frac{n\pi}{2}\right) \quad (4)$$

where C_n is a constant and Z_0 is the wave impedance $(= \omega\mu/k)$ at $x = 0$.

B. Unacceptable Solution

There exists the other solution whose asymptotic representation is

$$\Psi_n(x) \simeq C_n \frac{\sqrt{2Z_0}}{\sqrt[4]{b_n - \chi(x)}} \sin\left(\int_0^x k\sqrt{b_n - \chi(t)}\, dt - \frac{n\pi}{2}\right). \quad (5)$$

The $\Psi_n(x)$ is not an acceptable solution in the ideal waveguide because it grows up exponentially as x tends to infinity. Equation (4) expresses a standing wave, as shown in Fig. 2, and is decomposed into the incident wave $\Phi_n(x) - j\Psi_n(x)$ transmitting only the stored energy [4] with the Poynting power C_n^2 toward the x direction and the reflected wave $\Phi_n(x) + j\Psi_n(x)$ transmitting the same energy toward the opposite direction. These two waves give rise to a resonance in the transverse plane. When we regard this resonant system as a network, and choose a driving point of the network at $x = 0$, the odd and even numbers of n correspond to a resonance and antiresonance, respectively. According to the network theory [5], it is concluded that the field distribution is symmetric (antisymmetric) when an antiresonance (a resonance) occurs.

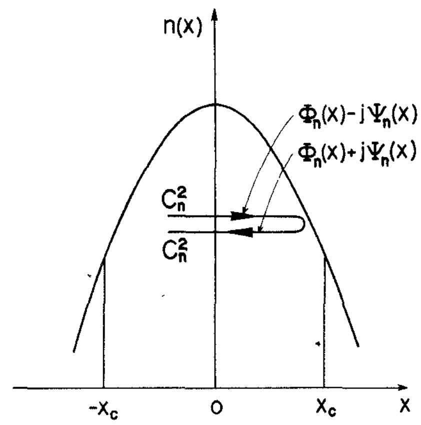

Fig. 2. Interpretation of the field resonance in the ideal waveguide.

$$\left.\begin{array}{l}\Phi_n(x) = \text{even} \\ \Psi_n(x) = \text{odd}\end{array}\right\}, \quad \text{for } n = 0,2,4,\cdots$$

$$\left.\begin{array}{l}\Phi_n(x) = \text{odd} \\ \Psi_n(x) = \text{even}\end{array}\right\}, \quad \text{for } n = 1,3,5,\cdots. \quad (6)$$

This is true even if the outer layer exists. Therefore (6) becomes important in the next section.

A set of $\Phi_n(x)$ and $\Psi_n(x)$ will be needed when we calculate the effect of the cladding. The function $\Psi_n(x)$ can uniquely be determined as follows, recognizing the function $\Phi_n(x)$.

1) Give $\Phi_n(x)$.

2) Calculate C_n with $\Phi_n(x)$. This is done by expanding (4) into a power series of x

$$\Phi_n(x) \simeq C_n \frac{\sqrt{2Z_0}}{\sqrt[4]{b_n}} \cos\left(\frac{n\pi}{2}\right) + C_n k\sqrt{2Z_0}\sqrt[4]{b_n} \sin\left(\frac{n\pi}{2}\right) x. \quad (7)$$

Thus, at $x = 0$,

$$C_n \simeq \begin{cases} \dfrac{\sqrt[4]{b_n}}{\sqrt{2Z_0}} \dfrac{\Phi_n(0)}{\cos\left(\frac{n\pi}{2}\right)}, & \text{for } n = 0,2,4,\cdots \\[2ex] \dfrac{1}{\sqrt{2Z_0}\, k\sqrt[4]{b_n}} \dfrac{\Phi_n'(0)}{\sin\left(\frac{n\pi}{2}\right)}, & \text{for } n = 1,3,5,\cdots. \end{cases} \quad (8)$$

3) Seek the solution $\Psi_n(x)$ that satisfies the power conservation law (Poynting theorem)

$$\Phi_n(x)\Psi_n'(x) - \Phi_n'(x)\Psi_n(x) = 2kC_n^2 Z_0 \quad (9)$$

and the even–odd condition (6).

Modes in a Practical Waveguide

We now consider the inhomogeneous wave propagation in the cladded inhomogeneous waveguide (Fig. 1). The propagation constants for this waveguide will change the form as

$$\beta_\nu = k\sqrt{1 - b_\nu} \qquad \nu = n + \Delta\nu_n, \qquad (n = 0,1,2,\cdots) \tag{10}$$

where the symbol β_ν or b_ν emphasizes the analytic continuation which is achieved by replacing n by ν in the functional form of β_n or b_n. The analytically continued functions $\Phi_\nu(x)$ and $\Psi_\nu(x)$ are, then,

$$\Phi_\nu(x) \simeq C_\nu \frac{\sqrt{2Z_0}}{\sqrt[4]{b_\nu - \chi(x)}} \cos\left(\int_0^x k\sqrt{b_\nu - \chi(t)}\, dt - \frac{\nu\pi}{2}\right)$$
$$\Psi_\nu(x) \simeq C_\nu \frac{\sqrt{2Z_0}}{\sqrt[4]{b_\nu - \chi(x)}} \sin\left(\int_0^x k\sqrt{b_\nu - \chi(t)}\, dt - \frac{\nu\pi}{2}\right). \tag{11}$$

The transverse mode function can be written, in the guiding region, as a linear combination of $\Phi_\nu(x)$ and $\Psi_\nu(x)$

$$\Phi_\nu(x)\cos\theta_\nu - \Psi_\nu(x)\sin\theta_\nu \simeq C_\nu \frac{\sqrt{2Z_0}}{\sqrt[4]{b_\nu - \chi(x)}} \cos\left(\int_0^x k\sqrt{b_\nu - \chi(t)}\, dt - \frac{\nu\pi}{2} + \theta_\nu\right). \tag{12}$$

This solution will still obey the "even–odd rule" for n, as mentioned before; (12) is an even (or an odd) function of x when n is even (or odd), and we see at once that $-\nu\pi/2 + \theta_\nu = -n\pi/2$, or

$$\theta_\nu = \frac{\pi}{2}\Delta\nu_n. \tag{13}$$

The continuous electric-field function of nth mode is now written as

$$\text{field function} = \begin{cases} \Phi_\nu(x)\cos\left(\frac{\pi}{2}\Delta\nu_n\right) - \Psi_\nu(x)\sin\left(\frac{\pi}{2}\Delta\nu_n\right), & \text{for } |x| < x_c \\ A_\nu \exp\left(-k\sqrt{\chi(x_c) - b_\nu}\,|x - x_c|\right), & \text{for } |x| > x_c \end{cases} \tag{14}$$

where

$$A_\nu \equiv \Phi_\nu(x_c)\cos\left(\frac{\pi}{2}\Delta\nu_n\right) - \Psi_\nu(x_c)\sin\left(\frac{\pi}{2}\Delta\nu_n\right).$$

The magnetic-field component H_z proportional to the derivative of (14) with respect to x, must also be continuous at $x = x_c$. Thus

$$K_{\Phi_\nu}\cos\left(\frac{\pi}{2}\Delta\nu_n\right) - K_{\Psi_\nu}\sin\left(\frac{\pi}{2}\Delta\nu_n\right) = 0 \tag{15}$$

where K_{Φ_ν} and K_{Ψ_ν} denote surface currents flowing toward the direction perpendicular to the x axis, at $x = x_c$; when the internal electric field is forced to be $\Phi_\nu(x)e^{-j\beta_\nu z}$ (or $\Psi_\nu(x)e^{-j\beta_\nu z}$), the surface current K_{Φ_ν} (or K_{Ψ_ν}) is induced in the presence of magnetic-field discontinuity. Equation (15) means that the total surface current vanishes at $x = x_c$. Thus we have

$$\Delta\nu_n = \left(\frac{2}{\pi}\right)\tan^{-1}\left(\frac{K_{\Phi_\nu}}{K_{\Psi_\nu}}\right) = \left(\frac{2}{\pi}\right)\tan^{-1}\left[\frac{\Phi_\nu'(x_c) + k\sqrt{\chi(x_c) - b_\nu}\,\Phi_\nu(x_c)}{\Psi_\nu'(x_c) + k\sqrt{\chi(x_c) - b_\nu}\,\Psi_\nu(x_c)}\right]. \tag{16}$$

Noting that $\Delta\nu_n$ is usually small compared to n, we approximate

$$\Delta\nu_n \simeq \left(\frac{2}{\pi}\right)\tan^{-1}\left[\frac{\Phi_n'(x_c) + k\sqrt{\chi(x_c) - b_n}\,\Phi_n(x_c)}{\Psi_n'(x_c) + k\sqrt{\chi(x_c) - b_n}\,\Psi_n(x_c)}\right]. \tag{17}$$

At this stage, the present approximate method for determining the propagation constants will be summarized.

1) Give the ideal waveguide solutions $\Phi_n(x)$ and β_n in analytic form.

2) Obtain the associated solution $\Psi_n(x)$ from (6), (8), and (9).

3) Substitute the specified function $\Phi_n(x)$ and the associated function $\Psi_n(x)$ just obtained into the right-hand side of (17), and calculate $\Delta\nu_n$.

4) In β_n, replace n by $\nu \equiv n + \Delta\nu_n$, and compute β_ν.

The four steps will be explained in detail when the method is applied to some cases of index profile.

Examples

A. Parabolic-Index Profile (TE Wave Propagation)

$$\chi(x) = \chi_0(x) \equiv (gx)^2$$

where g is the medium constant.

The validity of the present theory is firmly ascertained when the theory is applied to this case and when the results are compared to those obtained with a slight modification of the rigorous analysis [2].

Step 1: In this case, $\Phi_n(x)$ and b_n are well known [6].

$$\Phi_n(x) = \Phi_{n0}(x) \equiv H_n(\sqrt{2}\,x/S_0)e^{-x^2/2S_0^2}, \qquad S_0 = 1/\sqrt{kg}$$
$$b_n = b_{n0} \equiv (g/k)(2n + 1) \tag{18}$$

where $H_n(x)$ is the Hermite polynomial of nth degree, defined as

$$H_n(x) \equiv (-1)^n e^{x^2/2}(d^n/dx^n)(e^{-x^2/2}). \tag{19}$$

Note that (19) may often be denoted by $H_{en}(x)$.

Step 2: We first calculate C_n with (8) using the formulas $H_{2m}(0) = (-1)^m(2m-1)!!$, $H'_{2m+1}(0) = (-1)^m(2m+1)!!$, where $(2m-1)!! \equiv (2m-1)(2m-3)\cdots 1$ and $(-1)!! \equiv 1$.

$$C_n \simeq \begin{cases} (n-1)!!\sqrt[4]{2n+1}\sqrt{\frac{1}{2Z_0}\sqrt{\frac{g}{k}}}, & \text{for } n = 0,2,4,\cdots \\ \frac{n!!}{\sqrt[4]{2n+1}}\sqrt{\frac{1}{Z_0}\sqrt{\frac{g}{k}}}, & \text{for } n = 1,3,5,\cdots. \end{cases} \tag{20}$$

Second, to obtain the associated solution $\Psi_n(x)$, the Hermite function $h_n(x)$ [7] of the second kind defined below is introduced

$$h_n(x) \equiv (-1)^n e^{x^2/2}(d^n/dx^n)\left(e^{-x^2/2}\int_0^x e^{t^2/2}\,dt\right). \quad (21)$$

Equation (21) is another solution of Hermite's differential equation, which is chosen to be an odd or an even function of x according to whether n is even or odd, and may be obtained from the theory of parabolic cylinder functions [8], [9]. The $h_n(x)$ satisfies the same recurrence formulas with those for $H_n(x)$.[1] In addition, $h_n(x)$ is associated with $H_n(x)$ by

$$H_n(x)h_n'(x) - H_n'(x)h_n(x) = n!\, e^{x^2/2}. \quad (22)$$

It follows from (22) that the associated solution $\Psi_n(x)$ satisfying (6) and (9) is given by

$$\Psi_n(x) = \Psi_{n0}(x) \equiv \frac{Z_0}{n!}\sqrt{\frac{2k}{g}}\, C_n^2 h_n(\sqrt{2}\, x/S_0)e^{-x^2/2S_0^2}. \quad (23)$$

Step 3: The functions $\Phi_n(x)$ and $\Psi_n(x)$ are substituted into (17).

$$\Delta\nu_n \simeq \left(\frac{2}{\pi}\right)\tan^{-1}\left[\eta_n \frac{H_n'(\sqrt{2}\,\xi_c) + (\sqrt{\xi_c^2 - 2n - 1} - \xi_c)H_n(\sqrt{2}\,\xi_c)}{h_n'(\sqrt{2}\,\xi_c) + (\sqrt{\xi_c^2 - 2n - 1} - \xi_c)h(\sqrt{2}\,\xi_c)}\right] \quad (24)$$

where $\xi_c \equiv x_c/S_0$ and

$$\eta_n \equiv \frac{n!}{Z_0}\sqrt{\frac{g}{2k}}\frac{1}{C_n^2}. \quad (25)$$

The values of $-\Delta\nu_n$ versus x_c/S_0 are plotted in Fig. 3. Then, to avoid numerical errors, $h_n(\sqrt{2}\,\xi)$ is computed using the new function $N_n(\xi)$ defined in footnote 2.

Step 4: Calculation of the propagation constant is straightforward.

$$\beta_\nu = k\sqrt{1 - b_\nu} = k\sqrt{1 - (g/k)(2\nu + 1)}$$
$$= k\sqrt{1 - (g/k)(2n + 1 + 2\Delta\nu_n)}. \quad (26)$$

[1] $h_n''(x) - xh_n'(x) + nh_n(x) = 0$

$h_n'(x) = nh_{n-1}(x), \quad (n \geq 1)$

$h_{n+1}(x) - xh_n(x) + nh_{n-1}(x) = 0, \quad (n \geq 1)$

$h_0(x) = \int_0^x e^{t^2/2}\,dt$

$h_1(x) = x\int_0^x e^{t^2/2}\,dt - e^{x^2/2}.$

[2] $N_n(\xi) \equiv (\sqrt{2}\,\xi)^{n+1}e^{-\xi^2}h_n(\sqrt{2}\,\xi)/n!$

$N_n(\xi) = (2\xi^2/n)[N_{n-1}(\xi) - N_{n-2}(\xi)], \quad (n \geq 1)$

$N_{-1}(\xi) \equiv 1$

$N_n(\xi) \sim 1 + \frac{(n+1)(n+2)}{1!\,(2\xi)^2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2!\,(2\xi)^4} + \cdots$

$N_0(\xi) = 2\xi e^{-\xi^2}\int_0^\xi e^{t^2}\,dt$

$= \sum_{l=0}^{\infty} \frac{(-1)^l}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2l+1)}(2\xi^2)^{l+1}.$

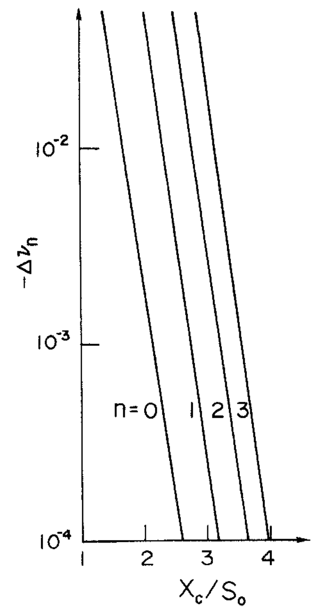

Fig. 3. A plot of $-\Delta\nu_n$ versus x_c/S_0 for a parabolic-index profile.

The rigorous analysis [2] yields the same results with (24) and (26), except that in [2], (25) is given by $\sqrt{\pi/2}$. Therefore,[3] the error in the present approximate analysis is about 11 percent when $n = 0$ and becomes less than 1 percent when n is larger than 2.

When $k_0 = 10^4$ mm^{-1}, $n_0 = 1.53$, $g = 3.23$ mm^{-1}, and $x_c = 12.6\ \mu$m ($x_c = 2.8S_0$; $S_0 = 4.5\ \mu$m), the four modes will be able to propagate in the waveguide. Then

$$\Delta\nu_0 \simeq -0.4 \times 10^{-4} \qquad \Delta\nu_1 \simeq -0.9 \times 10^{-3}$$
$$\Delta\nu_2 \simeq -0.9 \times 10^{-2} \qquad \Delta\nu_3 \simeq -0.8 \times 10^{-1}.$$

The values of the propagation constants increase due to the effect of the cladding. Such increments are, respectively, 1.3×10^{-4} ($n = 0$), 2.9×10^{-3} ($n = 1$), 2.9×10^{-2} ($n = 2$), and 2.6×10^{-1} ($n = 3$) in mm^{-1}.

B. Near-Parabolic-Index Profile (TE Wave Propagation)

$$\chi(x) = \chi_1(x) \equiv (gx)^2 + \alpha(gx)^4$$

where α, g are constants.

Let $\chi_0(x)$ and $\chi_1(x)$ be parabolic and near-parabolic functions, respectively. For simplicity, the difference between $\chi_1(x)$ and $\chi_0(x)$ is assumed to be biquadratic.

Step 1: When g/k is small compared with unity, $\Phi_n(x)$ and b_n are approximately expressed by [10]

$$\Phi_n(x) = \Phi_{n1}(x) \equiv [dy(x)/dx]^{-1/2} \cdot H_n[\sqrt{2}\, y(x)/S_0]e^{-y^2(x)/2S_0^2}$$

$$b_n = b_{n1} \simeq (g/k)(2n+1) + (g/k)^2\alpha(3/4)(2n^2 + 2n + 1) - (g/k)^3\alpha^2[(17/64)(2n+1)^3 + (67/64)(2n+1)] \quad (27)$$

[3] $\eta_0, \eta_1, \eta_2, \eta_3, \cdots$, are, respectively, 1.4142, 1.2247, 1.2649, 1.2472, 1.2571, 1.2508, $\cdots$, which tend to $\sqrt{\pi/2} = 1.2533$ rapidly as n increases.

where the subscript 1 in $\Phi_{n1}(x)$ and b_{n1} indicates that the index distribution is near parabolic [refer to the subscript 0 in (18) indicating the parabolic state], and $y(x)$ is a function of x approximately proportional to x. The implicit expression for $y(x)$ is approximately given by

$$\int_0^{y(x)} \sqrt{b_{n0} - \chi_0(y)}\, dy = \int_0^x \sqrt{b_{n1} - \chi_1(x)}\, dx \quad (28)$$

and the second-order solution of (28) is [10]

$$y(x) \simeq x + (g/k)\alpha S_0[(3/16)(2n + 1)(x/S_0) + (1/8)(x/S_0)^3]. \quad (29)$$

Step 2: To obtain C_n, $\Phi_{n1}(0) = H_n(0)/\sqrt{y'(0)}$ and $\Phi_{n1}'(0) = \sqrt{2y'(0)}\, H_n'(0)/S_0$ are substituted into (8). We see that $\sqrt[4]{b_{n1}}$ is equal to $\sqrt[4]{b_{n0}} \cdot \sqrt{y'(0)}$ within an accuracy of first order of g/k. Therefore, we obtain the same result with (20).

The $\Psi_n(x)$ can be obtained in a similar manner with (23); it is readily seen that

$$\Psi_n(x) = \Psi_{n1}(x) \equiv (1/\eta_n)[dy(x)/dx]^{-1/2} \cdot h_n[\sqrt{2}\, y(x)/S_0]e^{-y^2(x)/2S_0^2} \quad (30)$$

where η_n is the same as in (25).

Step 3: Substitute (27) and (30) into (17). Then we neglect the second derivative $y''(x)$, because $y(x)$ is approximately proportional to x. This is valid in the range $g/k \ll 1$. The detailed derivation for $\Delta\nu_n$ is given in the Appendix. The result is

$$\Delta\nu_n \simeq \left(\frac{2}{\pi}\right) \tan^{-1} \cdot \left[\frac{\Phi_{n0}'(y_c) + k\sqrt{\chi_0(y_c) - b_{n0}}\, \Phi_{n0}(y_c)}{\Psi_{n0}'(y_c) + k\sqrt{\chi_0(y_c) - b_{n0}}\, \Psi_{n0}(y_c)}\right] \quad (31)$$

where

$$y_c \simeq x_c + (g/k)\alpha S_0[(3/16)(2n + 1)(x_c/S_0) + (1/8)(x_c/S_0)^3]. \quad (32)$$

It is worth noting that (31) is equivalent to the formula (24) obtained for the parabolic-index profile only if x_c is replaced by y_c. The $\Delta\nu_n$ can be calculated using (32) and (24).

Step 4: The propagation constant is computed as follows:

$$\begin{aligned}\beta_\nu &= k\sqrt{1 - b_\nu}\\ b_\nu &= (g/k)(2\nu + 1) + (g/k)^2\alpha(3/4)(2\nu^2 + 2\nu + 1)\\ &\quad - (g/k)^3\alpha^2[(17/64)(2\nu + 1)^3 + (67/64)(2\nu + 1)]\\ \nu &= n + \Delta\nu_n. \end{aligned} \quad (33)$$

To see the effect of the fourth aberration term in propagation characteristics, we use the same parameters with those in the previous subsection. For $\alpha = 100$ and $n = 3$, y_c is 13.2 μm ($2.94S_0$). Thus, $\Delta\nu_3 \simeq -0.36 \times 10^{-1}$ and the increment of the propagation constant due to the effect of the cladding $\simeq 1.1 \times 10^{-1}$ mm^{-1}.

C. *Near-Parabolic-Index Profile (TM Wave Propagation)*

$$\chi(x) = \chi_1(x) \equiv (gx)^2 + \alpha(gx)^4$$

where α, g are constants.

This subsection includes a case of parabolic-index profile.

Step 1: For TM waves, the ideal waveguide solutions are given in [10]. The propagation constant is

$$\beta_n = k\sqrt{1 - b_n - (g/k)^2}. \quad (34)$$

The transverse mode-function (magnetic field) and b_n are given by

$$\begin{aligned} g &\to \bar{g} \equiv g\sqrt{1 + (6\alpha + 4)(g/k)^2}\\ S_0 &\to \bar{S}_0 \equiv S_0/\sqrt[4]{1 + (6\alpha + 4)(g/k)^2} \end{aligned} \quad (35)$$

in (27).

Step 2: The derivation of $\Psi_n(x)$ developed here does not depend on whether the field is of TE type or TM type. Therefore, the result is the same with (30), if the replacement (35) is taken into account.

Step 3: K_{Φ_ν} and K_{Ψ_ν} in (15) must be replaced by surface magnetic currents. However, the surface electric and magnetic currents are similar in form; the difference is negligible if g/k is small. The formula (17) is also applicable to this case when the first-order effect of the cladding is to be evaluated.

Step 4: The propagation constant can be computed as follows:

$$\begin{aligned}\beta_\nu &= k\sqrt{1 - b_\nu - (g/k)^2}\\ b_\nu &= (\bar{g}/k)(2\nu + 1) + (\bar{g}/k)^2\alpha(3/4)(2\nu^2 + 2\nu + 1)\\ &\quad - (\bar{g}/k)^3\alpha^2[(17/64)(2\nu + 1)^3 + (67/64)(2\nu + 1)]\\ \nu &= n + \Delta\nu_n. \end{aligned} \quad (36)$$

Note that $\Delta\nu_n$ has been obtained in Step 3 replacing ξ_c in (24) by $y_c/\bar{S}_0$ where

$$y_c \simeq x_c + (\bar{g}/k)\alpha\bar{S}_0[(3/16)(2n + 1)(x_c/\bar{S}_0) + (1/8)(x_c/\bar{S}_0)^3]. \quad (37)$$

The modification previously described has completed the application of the present theory to the TM wave propagation. A more accurate analysis may be developed by applying the second-order theory of the TM wave propagation [10] to this case.

Conclusion

An approximate theory has been developed for the analysis of the wave propagation in cladded inhomogeneous dielectric waveguides. It has been shown that the propagation constants can be obtained from the ideal waveguide solutions, which are the analytic solutions in the absence of cladding. Some applications have been presented. For a parabolic-index profile, the results of the rigorous analysis have been shown by which the present theory has been verified.

Appendix

Derivation of (31) is as follows:

$$\begin{aligned}\Phi_{n1}'(x_c) &+ k\sqrt{\chi_1(x_c) - b_{n1}}\,\Phi_{n1}(x_c)\\ &= [dy/dx]^{1/2}\{\Phi_{n0}'(y_c) - [y''(x_c)/2y'^2(x_c)]\Phi_{n0}(y_c)\\ &\quad + (dy/dx)^{-1}k\sqrt{\chi_1(x_c) - b_{n1}}\,\Phi_{n0}(y_c)\}\\ &= [dy/dx]^{1/2}\{\Phi_{n0}'(y_c) + k\sqrt{\chi_0(y_c) - b_{n0}}\,\Phi_{n0}(y_c)\\ &\quad - [y''(x_c)/2y'^2(x_c)]\Phi_{n0}(y_c)\}\\ &\simeq [dy/dx]^{1/2}\{\Phi_{n0}'(y_c) + k\sqrt{\chi_0(y_c) - b_{n0}}\,\Phi_{n0}(y_c)\}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\Psi_{n1}'(x_c) &+ k\sqrt{\chi_1(x_c) - b_{n1}}\,\Psi_{n1}(x_c)\\ &\simeq [dy/dx]^{1/2}\{\Psi_{n0}'(y_c) + k\sqrt{\chi_0(y_c) - b_{n0}}\,\Psi_{n0}(y_c)\}.\end{aligned}$$

Substituting these results into (17), we obtain (31).

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Fringing-Field Effects in Edge-Guided Wave Devices

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***Abstract*—An equivalent model is presented for evaluating the fringing-field effects in edge-guided waves (EGW) propagating along ferrite microstrip circuits. It is based on the approximate model developed by Getsinger for nonferromagnetic microstrip circuits. Fringing-field effects are characterized by a fringing-field parameter b/b' whose numerical value is determined by experiment. Measurements are made on EGW resonators of various shapes for different values of the applied magnetic bias.**

Finally, the fringing-field parameter is used to evaluate the ratio between the reactive power stored in the fringing fields and the RF power in the ferrite under the strip conductor in a disk resonator.

I. Introduction

Fringing-field effects in ferrite microstrips have been studied by a number of authors [1]–[3]. Similar to what has been done for microstrips on a nonmagnetic substrate, they are accounted for by introducing a "magnetic" filling factor, which reduces the numerical value of the magnetic permeability of the substrate. Therefore, in ferrite microstrip lines both the dielectric permittivity and the magnetic permeability of the substrate's material have "effective" values. In general they are given in the form

$$\varepsilon_{\text{eff}} = 1 + q_e(\varepsilon_r - 1) \tag{1}$$

$$\mu_{\text{eff}} = \left[1 + q_m\left(\frac{1}{\mu_r} - 1\right)\right]^{-1} \tag{2}$$

where q_e and q_m are the electric and magnetic filling factors, ε_r and μ_r are the relative dielectric permittivity and magnetic permeability of the substrate. The two quantities q_e and q_m depend upon the geometry of the microstrip circuit and are in general different from each other [4].

The expressions (1) and (2) are valid under the hypothesis that the substrate is characterized by scalar constitutive

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